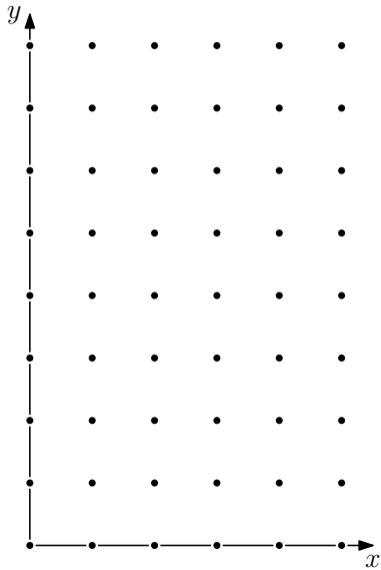
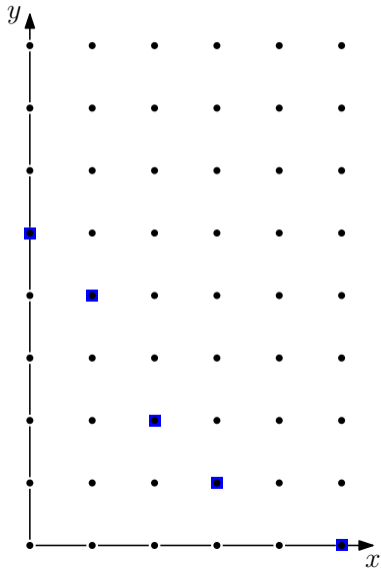


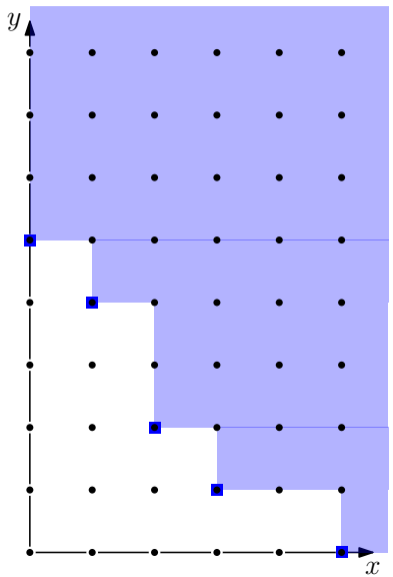
# FGLM – Time for a dictionary



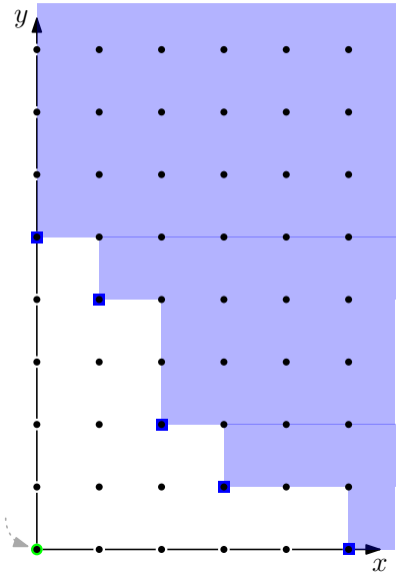
# FGLM – Time for a dictionary



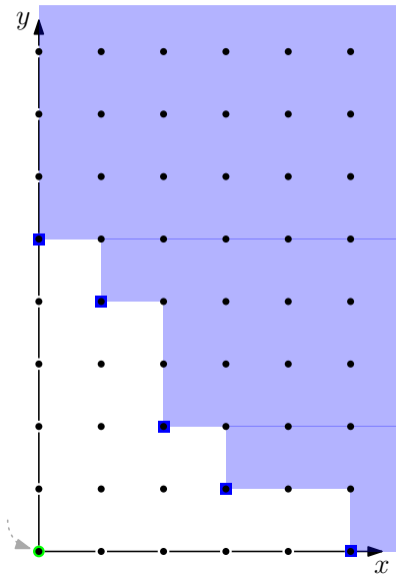
# FGLM – Time for a dictionary



# FGLM – Time for a dictionary

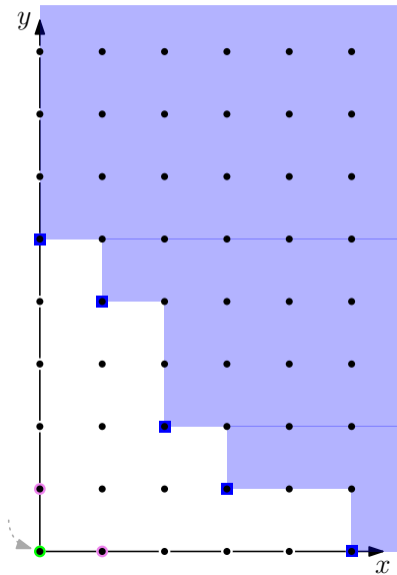


# FGLM – Time for a dictionary



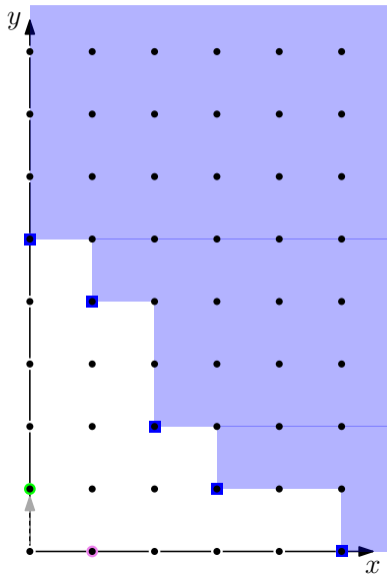
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1

# FGLM – Time for a dictionary



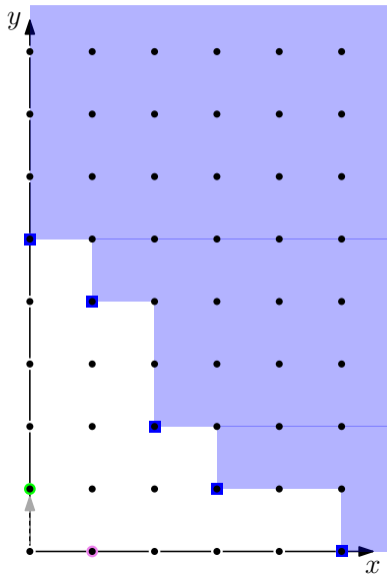
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1

# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1

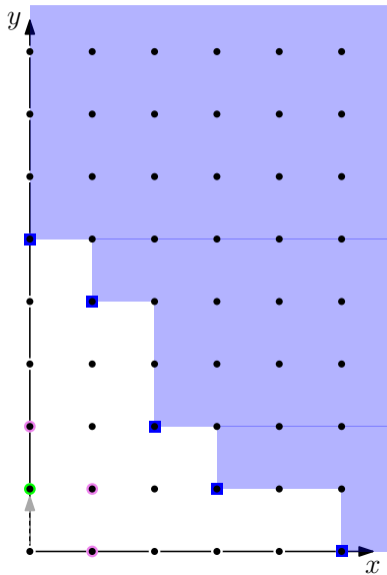
# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$

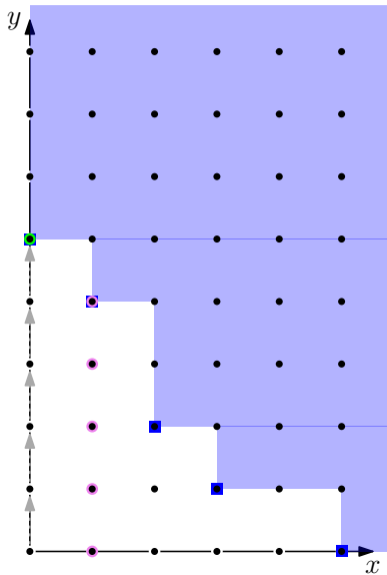


# FGLM – Time for a dictionary



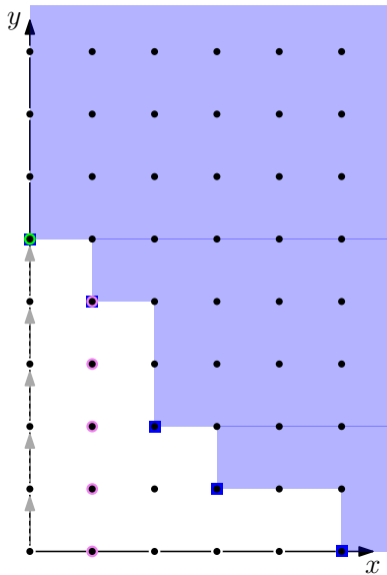
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$

# FGLM – Time for a dictionary



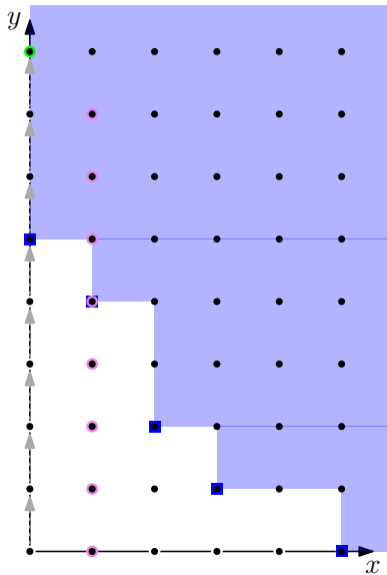
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$

# FGLM – Time for a dictionary



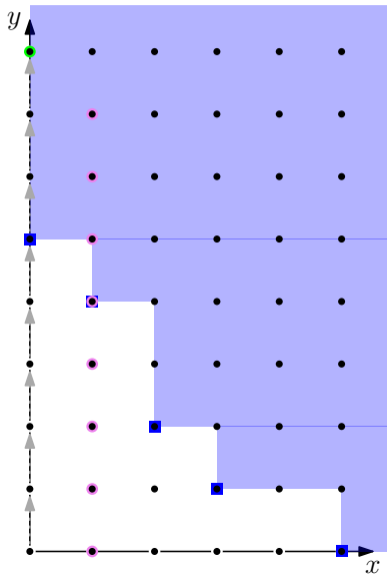
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$

# FGLM – Time for a dictionary



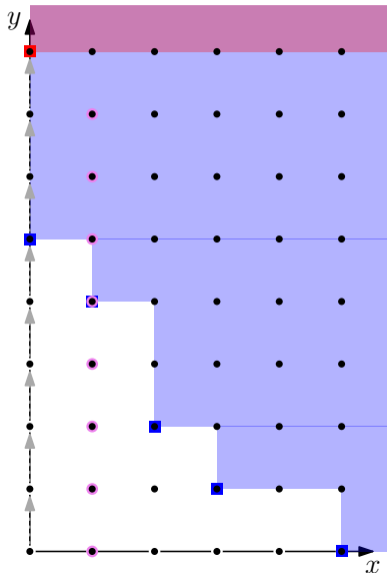
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$

# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0

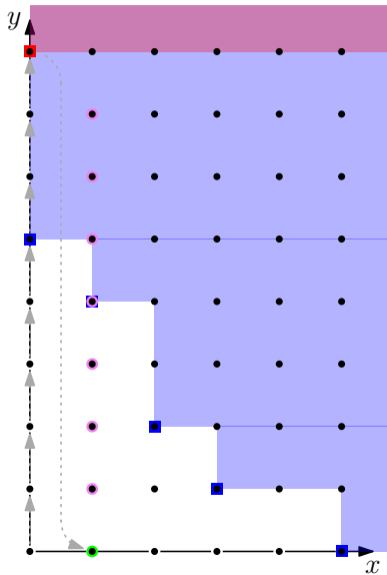
# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0

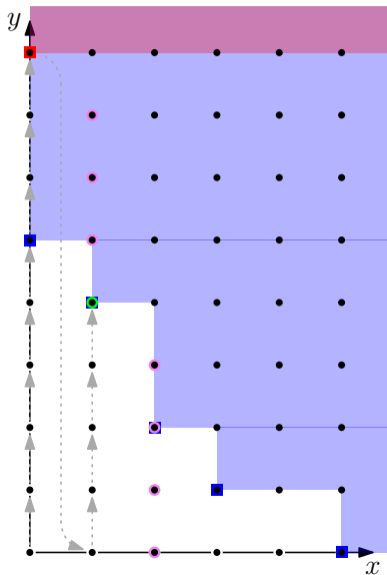
$g_0$

# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$

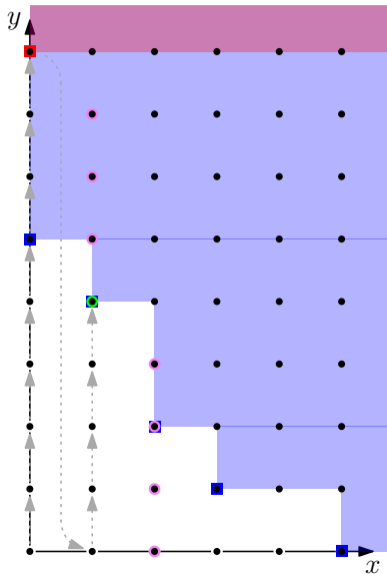
# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$



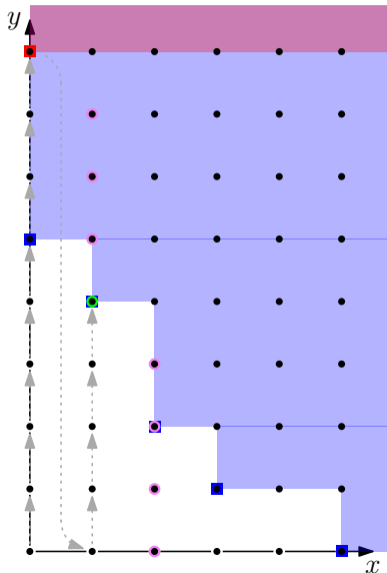
# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$

-1

# FGLM – Time for a dictionary

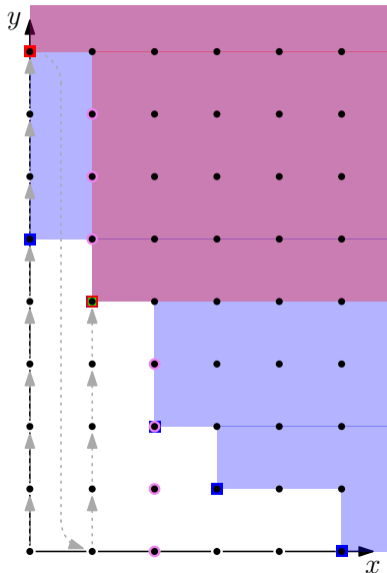


$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$

$+1$  (red) and  $-1$  (blue) arrows indicate the update of the dictionary. The  $+1$  arrow points from the  $y^6$  row to the  $xy^4$  row. The  $-1$  arrow points from the  $y^6$  row to the  $xy^4$  row.

$g_1$  (red) is indicated at the bottom left.

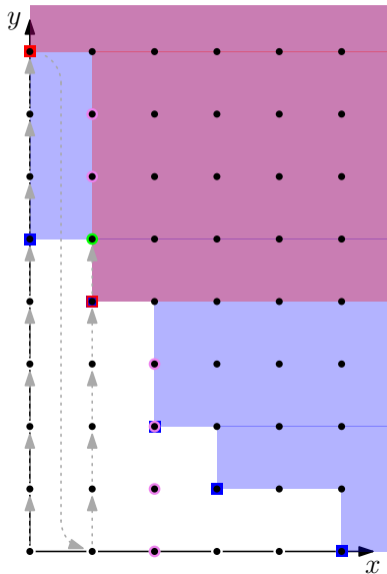
# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$

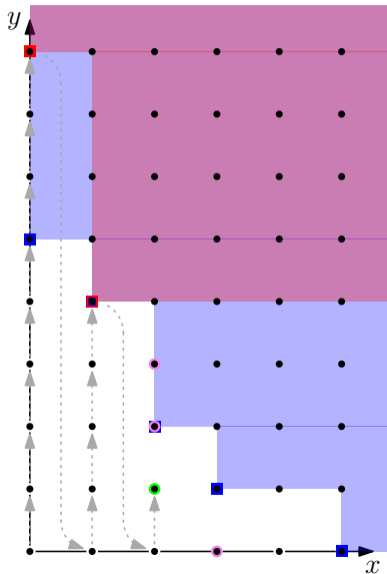
$+1$  (red) and  $-1$  (blue) arrows indicate the update of the leading term in the dictionary. The  $g_1$  (red) arrow points to the new leading term  $xy^4$ .

# FGLM – Time for a dictionary



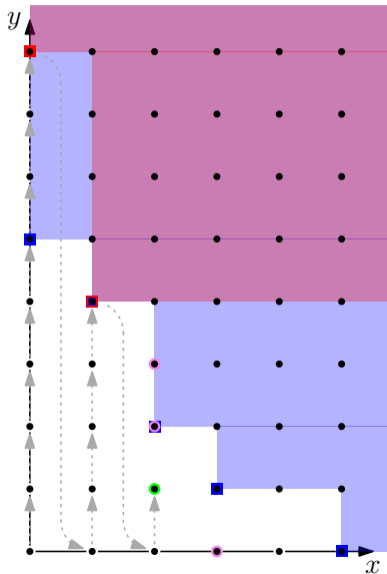
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$

# FGLM – Time for a dictionary



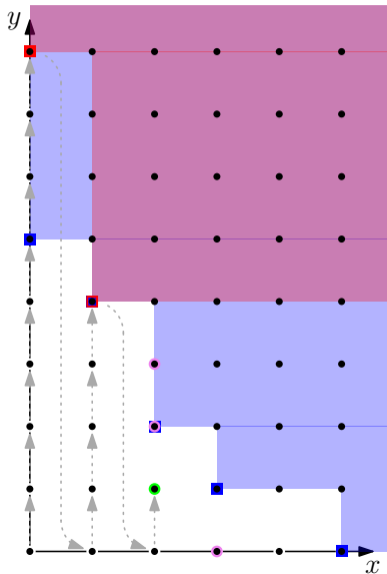
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$
$\vdots$	$\vdots$
$x^2$	$x^2$
$x^2y$	$x^2y$

# FGLM – Time for a dictionary



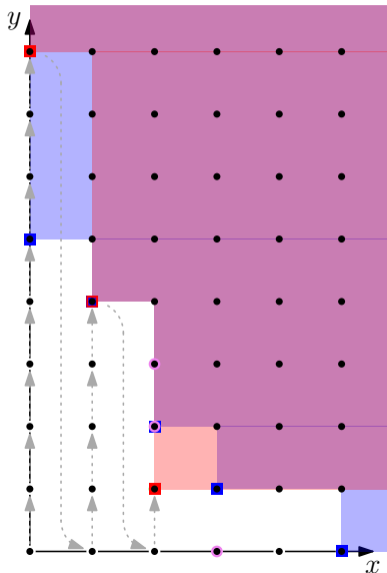
$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
y	y
⋮	⋮
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$ <span style="float: right;">+1</span>
$y^7$	$x^4 - x^3$ <span style="float: right;">-1</span>
$y^8$	0
x	x
⋮	⋮
$xy^4$	$-x^2y - x^4 + x^3$
⋮	⋮
$x^2$	$x^2$
$x^2y$	$x^2y$

# FGLM – Time for a dictionary



	$\prec_{\text{new}}$	$\prec_{\text{old}}$
	1	1
	$y$	$y$
	$\vdots$	$\vdots$
	$y^5$	$x^2y + x^4 - 2x^3 + x^2$
-1	$y^6$	$x^2y + x^4 - x^3$
+1	$y^7$	$x^4 - x^3$
	$y^8$	0
	$x$	$x$
	$\vdots$	$\vdots$
	$xy^4$	$-x^2y - x^4 + x^3$
	$\vdots$	$\vdots$
	$x^2$	$x^2$
$g_2$	$x^2y$	$x^2y$

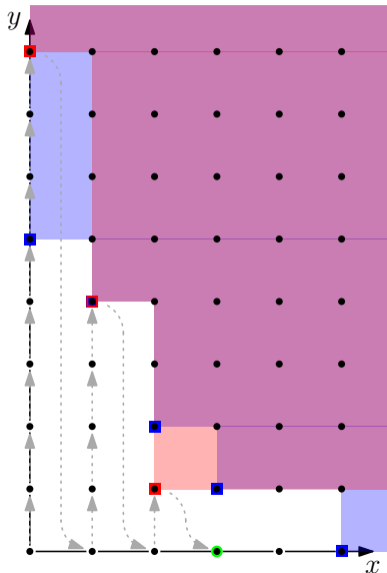
# FGLM – Time for a dictionary



	$\prec_{\text{new}}$	$\prec_{\text{old}}$	
	1	1	
	$y$	$y$	
	$\vdots$	$\vdots$	
	$y^5$	$x^2y + x^4 - 2x^3 + x^2$	
-1	$y^6$	$x^2y + x^4 - x^3$	+1
+1	$y^7$	$x^4 - x^3$	-1
	$y^8$	0	
	$x$	$x$	
	$\vdots$	$\vdots$	
	$xy^4$	$-x^2y - x^4 + x^3$	
	$\vdots$	$\vdots$	
	$x^2$	$x^2$	
$g_2$	$x^2y$	$x^2y$	

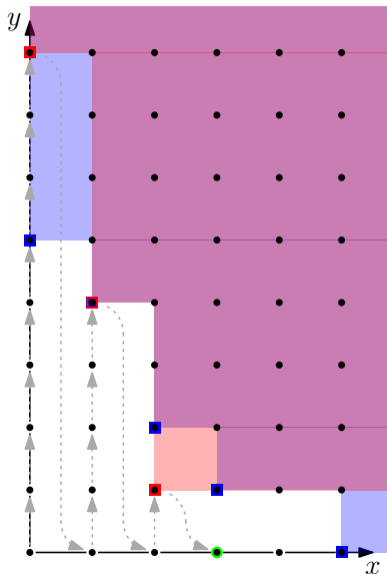


# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$
$\vdots$	$\vdots$
$x^2$	$x^2$
$x^2y$	$x^2y$
$x^3$	$x^3$

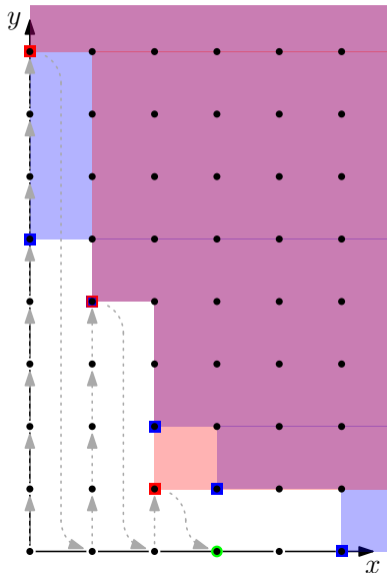
# FGLM – Time for a dictionary



$\prec_{\text{new}}$	$\prec_{\text{old}}$
1	1
$y$	$y$
$\vdots$	$\vdots$
$y^5$	$x^2y + x^4 - 2x^3 + x^2$
$y^6$	$x^2y + x^4 - x^3$
$y^7$	$x^4 - x^3$
$y^8$	0
$x$	$x$
$\vdots$	$\vdots$
$xy^4$	$-x^2y - x^4 + x^3$
$\vdots$	$\vdots$
$x^2$	$x^2$
$x^2y$	$x^2y$
$x^3$	$x^3$

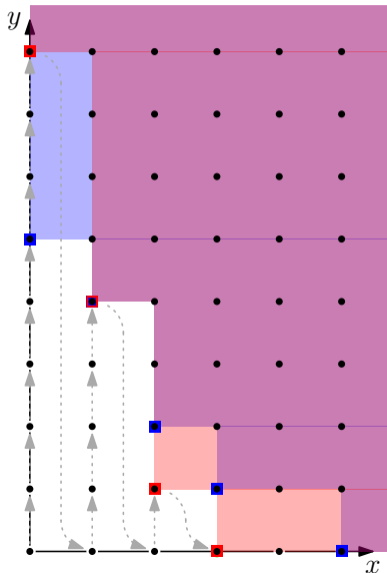
Annotations: A bracket on the right side groups the rows for  $y^5$ ,  $y^6$ ,  $y^7$ , and  $y^8$  with a '-1' label. Another bracket groups the rows for  $x^2$ ,  $x^2y$ , and  $x^3$  with a '+1' label. Arrows point from the  $x^2$  and  $x^3$  entries in the 'old' column to the right side of the table.

# FGLM – Time for a dictionary



	$\prec_{\text{new}}$	$\prec_{\text{old}}$	
	1	1	
	$y$	$y$	
	$\vdots$	$\vdots$	
+1	$y^5$	$x^2y + x^4 - 2x^3 + x^2$	-1
-1	$y^6$	$x^2y + x^4 - x^3$	+1
	$y^7$	$x^4 - x^3$	
	$y^8$	0	
	$x$	$x$	
	$\vdots$	$\vdots$	
	$xy^4$	$-x^2y - x^4 + x^3$	
	$\vdots$	$\vdots$	
-1	$x^2$	$x^2$	+1
	$x^2y$	$x^2y$	
$g_3$	$x^3$	$x^3$	

# FGLM – Time for a dictionary



	$\prec_{\text{new}}$	$\prec_{\text{old}}$	
	1	1	
	y	y	
	⋮	⋮	
+1	$y^5$	$x^2y + x^4 - 2x^3 + x^2$	-1
-1	$y^6$	$x^2y + x^4 - x^3$	+1
	$y^7$	$x^4 - x^3$	
	$y^8$	0	
	x	x	
	⋮	⋮	
	$xy^4$	$-x^2y - x^4 + x^3$	
	⋮	⋮	
-1	$x^2$	$x^2$	+1
	$x^2y$	$x^2y$	
$g_3$	$x^3$	$x^3$	

## FGLM – Got second base?

$G_{new}$  (lex  $x > y$ )

---

$$x^3 - x^2 - y^6 + y^5$$

$$x^2y + y^7 - y^6$$

$$xy^4 + y^6$$

$$y^8$$

$G_{old}$  (lex  $x < y$ )

---

$$y^5 - x^2y - x^4 + 2x^3 - x^2$$

$$xy^4 + x^2y + x^4 - x^3$$

$$x^2y^2 - x^4 + x^3$$

$$x^3y$$

$$x^5 - x^4$$

## FGLM – This is the *real* stuff

**Input:**  $\prec_{\text{new}}, G_{\text{old}}, \prec_{\text{old}}$

**Output:**  $G_{\text{new}}$

dict =  $\emptyset$

$G_{\text{new}} = \emptyset$

next\_monoms =  $\{1\}$

**while** next\_monoms  $\neq \emptyset$  **do**

    monom =  $\min_{\prec_{\text{new}}} \{\text{next\_monoms}\}$

    next\_monoms = next\_monoms  $\setminus \{\text{monom}\}$

**if**  $\nexists g \in G_{\text{new}}$  such that  $\text{LM}(g) \mid \text{monom}$  **then**     // still within staircase

        reduced\_monom = full reduction of monom by  $G_{\text{old}}$      // requires  $\prec_{\text{old}}$

**if** reduced\_monom +  $\sum_{\nu \in \text{dict}} \omega_{\nu} \text{value}(\nu) = 0$  for some  $\omega_{\nu} \in \mathbb{F}$  **then**

$G_{\text{new}} = G_{\text{new}} \cup \{\text{monom} + \sum_{\nu \in \text{dict}} \omega_{\nu} \text{key}(\nu)\}$

**else**

            dict = dict  $\cup$  (monom : reduced\_monom)

            next\_monoms = next\_monoms  $\cup \{x_i \cdot \text{monom} \mid i \in \{0, \dots, n-1\}\}$

**return**  $G_{\text{new}}$